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Existence Theorems in Conditional Gauss Variational Problems

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§ 1. Introduction

Let Ω be a locally compact Hausdorff space and G be a Borel measurable function on $\Omega\times\Omega$ which takes values in $(-\infty, \infty]$. We assume that $G(u, v) = G(v, u)$ for all $u, v \in \Omega$ and G is bounded below on every compact set. Such a function G is called a kernel. A non-negative Radon measure μ with compact support $S\mu$ will be called simply a measure. Denote by $M^+(\Omega)$ the totality of measures on Ω and put $M^+(S\mu) = \{\nu \in M^+(\Omega);\ S\nu \subset S\mu\}$ for $\mu \in M^+(\Omega)$. Given measures μ, ν on Ω , we define $G(u, \mu)$ and (ν, μ) by

$$G(u, \mu) = \int G(u, v)d\mu(v),$$

$$(\nu, \mu) = \int G(u, \mu)d\nu(u),$$

and call them the potential of μ and the mutual energy of μ and ν respectively. We call (μ, μ) simply the energy of μ. Denote by E the set of measures with finite energy.

We shall say that a property holds n.e. (= nearly everywhere) on a set $B \subset \Omega$ if it holds on B' such that $B' \subset B$, $\mu(K) = 0$ for all compact sets $K \subset B - B'$ and $\mu \in E$. A kernel G is termed to be of positive type or positive semi-definite if

$$(\mu - \nu, \mu - \nu) = (\mu, \mu) + (\nu, \nu) - 2(\nu, \mu) \geqq 0$$

for all $\mu, \nu \in E$. In case g and h are extended real-valued functions on Ω which are μ-summable for all $\mu \in E$, we set

$$g(u) - h(u) = 0$$

at points u where $g(u) = h(u) = \infty$ or $g(u) = h(u) = -\infty$.

Let B be a set in Ω which is measurable with respect to every $\mu \in E$ and satisfies the condition that $E'_B \neq \{0\}$, where

$$E'_B = \{\mu \in E;\ \mu(\Omega - B) = 0\}.$$

Let f and g_k, $k = 1, \cdots, n$, be real-valued functions on B which are μ-summable for every $\mu \in E'_B$ and $\{c_k;\ k = 1, \cdots, n\}$ be a set of real numbers. For simplicity, we shall write

$$\int f d\mu = < f, \mu >.$$

A mutual energy (μ, ν) can be written as $< G(\cdot\ , \mu), \nu >$.

We shall consider the problem of minimizing the expression (Gauss integral)

$$I(\mu) = (\mu, \mu) - 2 < f, \mu >$$

subject to $\mu \in E'_B$ and $< g_k, \mu > = c_k$ for each k. This is called the conditional Gauss variational problem. Let A be the transformation from E'_B into the n-dimensional Euclidean space R^n defined by

$$A\mu = (< g_1, \mu >, \cdots, < g_n, \mu >)$$

and let $z_o = (c_1, \cdots, c_n)$. Let us put

$$V = \inf\{I(\mu);\ \mu \in S\},$$
$$S = \{\mu \in E'_B;\ A\mu = z_o\},$$
$$S_o = \{\mu \in S;\ V = I(\mu)\}.$$

In case S_o is nonempty, V is finite by our assumption for f.

Let $\mu^* \in E'_B$. $\{g_k;\ k = 1, \cdots, n\}$ is called μ^*-independent if there exists a set $\{\mu_k;\ k = 1, \cdots, n\}$ in $M^+(S\mu^*)$ such that $\mu^* - \mu_k \in M^+(S\mu^*)$ for each k and $\det(< g_j, \mu_k >) \neq 0$, where $\det(a_{ij})$ means the determinant of a matrix (a_{ij}). The set $\{\mu_k;\ k = 1, \cdots, n\}$ is called a system of components of μ^*.

The following existence theorem was established by M. Ohtsuka ([1], p. 213, Theorem 2. 1).

Theorem 1. Assume that $\mu^* \in S_0$ and that $\{g_k; k = 1, \cdots, n\}$ is μ^*-independent. Let $\{\mu_k; k = 1, \cdots, n\}$ be a system of components of μ^* and $\{r_j; j = 1, \cdots, n\}$ be the solution of the equations

$$(1) \qquad \sum_{j=1}^{n} r_j < g_j, \mu_k > = < G(\cdot, \mu^*) - f, \mu_k >.$$

Then it is valid that

$$(2) \qquad G(\cdot, \mu^*) \geqq f + \sum_{k=1}^{n} r_k g_k \quad \text{n.e. on } B,$$

$$(3) \qquad G(\cdot, \mu^*) \leqq f + \sum_{k=1}^{n} r_k g_k \quad \mu_k\text{-a.e.}$$

for each k.

The aim of this paper is to study the roles of μ^*-independence in the conditional Gauss variational problem. We improve Theorem 1 and related results in [1]. For applications of our results and the existence of a measure of S_0, we refer to the forthcoming paper [2].

§ 2. Existence theorems

A system of components $\{\mu_k; k = 1, \cdots, n\}$ of μ is called to be full if $\mu = \sum_{k=1}^{n} \mu_k$. In this case we say that μ has a full system of components.

For $\mu \in E$, we define $C[\mu]$ by

$$C[\mu] = \{\nu \in M^+(S\mu); \mu - \nu \in M^+(S\mu)\}.$$

It is shown by the symmetry of G that $C[\mu]$ is a convex subset of E.

Let $Q(\mu)$ be the convex cone generated by $A(C[\mu])$, i.e., $z \in Q(\mu)$ if and only if there exist $\nu \in C[\mu]$ and a non-negative number t such that $z = tA\nu$. Denote by F^o the interior of a set F in R^n and by $((x, y))_2$ the usual inner product of $x, y \in R^n$.

We have

Lemma 1. Let $\mu^* \in S$. If $\{g_k;\ k = 1, \cdots, n\}$ is μ^*-independent, then $z_o \in Q(\mu^*)^o$.

Proof. Supposing the contrary, we see that z_o is a boundary point of $Q(\mu^*)$. There exists a nonzero $w = (r_1, \cdots, r_n) \in R^n$ by a well-known separation theorem such that

$$((z_o, w))_2 = 0 \leqq ((z, w))_2$$

for all $z \in Q(\mu^*)$. Let $\{\mu_k;\ k = 1, \cdots, n\}$ be a system of components of μ^*. From $\mu_k \in C[\mu^*]$ and $\mu^* - \mu_k \in C[\mu^*]$ for each k, it follows that

$$0 \leqq ((A\mu_k, w))_2 \leqq ((A\mu^*, w))_2 = ((z_o, w))_2 = 0,$$

so that

$$0 = ((A\mu_k, w))_2 = \sum_{j=1}^{n} r_j < g_j, \mu_k >.$$

Since $\det(< g_j, \mu_k >) \neq 0$, we conclude that $r_j = 0$ for each j. This is a contradiction. Therefore $z_o \in Q(\mu^*)^o$.

Lemma 2. Let $\mu^* \in S$ and assume that $\{g_k;\ k = 1, \cdots, n\}$ is μ^*-independent. Then μ^* has a full system of components.

Proof. Since $z_o \in Q(\mu^*)^o$ by Lemma 1, we can find a set $\{\nu_k;\ k = 1, \cdots, n\}$ of measures of $C[\mu^*]$ and a set $\{s_k;\ k = 1, \cdots, n\}$ of strictly positive numbers such that $\{A\nu_k;\ k = 1, \cdots, n\}$ is linearly independent and

$$z_o = \sum_{k=1}^{n} s_k A\nu_k.$$

In case $s_o = \sum_{k=1}^{n} s_k \leqq 1$, we have $\bar{\mu} = \sum_{k=1}^{n} s_k\nu_k \in S$ and $\nu = \mu^* - \bar{\mu} \in M^+(S\mu^*)$. Choosing $\mu_k^* = s_k\nu_k + \nu/n$ for each k, we see that $\{\mu_k^*; k = 1, \cdots, n\}$ is a full system of components of μ^*. In case $s_o > 1$, let us put $t_k = s_k/s_o$ for each k and consider $\mu_o = \sum_{k=1}^{n} t_k\nu_k$. Then $\mu_o \in C[\mu^*]$ and $A\mu_o = z_o/s_o$. Taking $\nu_o = \mu^* - \mu_o$ and $\mu_k^* = t_k\nu_k + \nu_o/n$, we can show that $\{\mu_k^*; k = 1, \cdots, n\}$ is a full system of components of μ^*.

For $\mu^* \in S_o$, we denote by $S_o^*(\mu^*)$ the set of points $w = (r_1, \cdots, r_n)$ which satisfy the following relations (4) and (5):

$$G(\cdot, \mu^*) \geqq f + \sum_{k=1}^{n} r_k g_k \quad \text{n.e. on } B, \tag{4}$$

$$G(\cdot, \mu^*) \leqq f + \sum_{k=1}^{n} r_k g_k \quad \mu^*\text{-a.e.} \tag{5}$$

If $w \in S_o^*(\mu^*)$, then it is valid that

$$< G(\cdot, \mu^*) - f, \mu^* > = ((z_o, w))_2, \tag{6}$$

$$V = I(\mu^*) = ((z_o, w))_2 - < f, \mu^* > = 2((z_o, w))_2 - (\mu^*, \mu^*). \tag{7}$$

It is easily seen that (4) and (6) imply (5).

We shall prove

Theorem 2. Assume that $\mu^* \in S_o$ and that $\{g_k; k = 1, \cdots, n\}$ is μ^*-independent. Then $S_o^*(\mu^*)$ consists of only one point $\bar{w}(\mu^*) = (r_1, \cdots, r_n)$. If $\{\mu_k; k = 1, \cdots, n\}$ is a system of components of μ^*, then $\{r_j; j = 1, \cdots, n\}$ is the solution of the equations

$$\sum_{j=1}^{n} r_j < g_j, \mu_k > = < G(\cdot, \mu^*) - f, \mu_k >. \tag{8}$$

Proof. Let $\{\mu_k^*;\ k = 1, \cdots, n\}$ be a full system of components of μ^* and define $\{r_j;\ j = 1, \cdots, n\}$ by

$$\sum_{j=1}^{n} r_j < g_j, \mu_k^* > = < G(\cdot, \mu^*) - f, \mu_k^* >.$$

It follows from Theorem 1 that

$$G(\cdot, \mu^*) \geqq f + \sum_{j=1}^{n} r_j g_j \quad \text{n.e. on } B, \tag{9}$$

$$G(\cdot, \mu^*) \leqq f + \sum_{j=1}^{n} r_j g_j \quad \mu^*\text{-a.e.}, \tag{10}$$

since $\mu^* = \sum_{k=1}^{n} \mu_k^*$. Therefore $\bar{w} = (r_1, \cdots, r_n) \in S_o^*(\mu^*)$. If $w = (s_1, \cdots, s_n) \in S_o^*(\mu^*)$, then

$$\sum_{j=1}^{n} s_j < g_j, \mu_k^* > = < G(\cdot, \mu^*) - f, \mu_k^* > = \sum_{j=1}^{n} r_j < g_j, \mu_k^* >,$$

so that

$$\sum_{j=1}^{n} (s_j - r_j) < g_j, \mu_k^* > = 0.$$

Since $\det(< g_j, \mu_k^* >) \neq 0$, we have $w = \bar{w}$. Namely $S_o^*(\mu^*)$ consists of only one point. Let $\{\mu_k;\ k = 1, \cdots, n\}$ be a system of components of μ^*. Then we have (8) by (9) and (10). This completes the proof.

Theorem 3. Assume that E_B' is convex and that $\mu^* \in S_o$. If $\{g_k;\ k = 1, \cdots, n\}$ is μ-independent for some $\mu \in S$, then $S_o^*(\mu^*)$ is nonempty.

Proof. From our assumption that E_B' is convex, it follows that S is convex. It is shown that

$$< G(\cdot, \mu^*) - f, \mu^* > \leqq < G(\cdot, \mu^*) - f, \nu >$$

for all $\nu \in S$. Writing $g_o = G(\cdot, \mu^*) - f$, we have

$$M = \langle g_o, \mu^* \rangle = \min\{\langle g_o, \nu \rangle; \nu \in S\},$$

$$V = \langle g_o, \mu^* \rangle - \langle f, \mu^* \rangle.$$

Our assumption that $\{g_k; k = 1, \cdots, n\}$ is μ-independent for some $\mu \in S$ is equivalent to the condition $z_o \in A(E_B')^o$. By means of a duality theorem in semi-infinite programs, we see that there exists $\bar{w} \in R^n$ such that

$$((A\nu, \bar{w}))_2 \leqq \langle g_o, \nu \rangle$$

for all $\nu \in E_B'$ and $M = ((z_o, \bar{w}))_2$. It is shown that $\bar{w} \in S_o^*(\mu^*)$.

E_B' is convex if and only if (μ, ν) is finite for all $\mu, \nu \in E_B'$. It is clear that E_B' is convex whenever G is of positive type or G is bounded on $B \times B$.

Theorem 4. Assume that G is of positive type and that μ^* and ν^* are elements of S_o. Then it is valid that $S_o^*(\mu^*) = S_o^*(\nu^*)$.

Proof. From $I(\mu^*) = I(\nu^*) = V$ and $(\mu^* + \nu^*)/2 \in S$, it follows that

$$I(\mu^*) \leqq I((\mu^* + \nu^*)/2) = I(\mu^*) - (\mu^* - \nu^*, \mu^* - \nu^*)/4,$$

and hence $(\mu^* - \nu^*, \mu^* - \nu^*) \leqq 0$. Since G is of positive type, we have $(\mu^* - \nu^*, \mu^* - \nu^*) = 0$ and $G(\cdot, \mu^*) = G(\cdot, \nu^*)$ n.e. in Ω. Consequently $(\mu^*, \mu^*) = (\mu^*, \nu^*) = (\nu^*, \nu^*)$ and $\langle f, \mu^* \rangle = \langle f, \nu^* \rangle$. Assume that $\bar{w} = (r_1, \cdots, r_n) \in S_o^*(\mu^*)$. By the above observation, we see that

$$G(\cdot, \nu^*) - f = G(\cdot, \mu^*) - f \geqq \sum_{k=1}^{n} r_k g_k \quad \text{n.e. on } B,$$

$$\langle G(\cdot, \nu^*) - f, \nu^* \rangle = \langle G(\cdot, \mu^*) - f, \mu^* \rangle = \sum_{k=1}^{n} r_k c_k$$

$$= \sum_{k=1}^{n} r_k \langle g_k, \nu^* \rangle = \langle \sum_{k=1}^{n} r_k g_k, \nu^* \rangle,$$

and hence

$$G(\cdot, \nu^*) - f = \sum_{k=1}^{n} r_k g_k \quad \nu^*\text{-a.e.},$$

so that $\bar{w} \in S_0^*(\nu^*)$. Therefore $S_0^*(\mu^*) \subset S_0^*(\nu^*)$. Since the discussion is symmetric, we have $S_0^*(\nu^*) \subset S_0^*(\mu^*)$ and hence $S_0^*(\mu^*) = S_0^*(\nu^*)$.

Corollary. Assume that G is of positive type and let μ^* and ν^* be elements of S_0. If $\{g_k; k = 1, \cdots, n\}$ is μ^*-independent and ν^*-independent, then it is valid that $\bar{w}(\mu^*) = \bar{w}(\nu^*)$.

We observe that Theorem 4 and its corollary are not always valid if G is not of positive type.

References

[1] M. Ohtsuka: On potentials in locally compact spaces, J. Sci. Hiroshima Univ. Ser. A-I Math., 25 (1961), 135 - 352.

[2] M. Yamasaki: Semi-infinite programs and conditional Gauss variational problems, Hiroshima Math. J., 1 (1971).